Algebra and Number Theory
Team Oral Test

1. Let E/F be a field extension. Let A be an $m \times m$ matrix with entries in E such that $tr(A^n)$ belongs to F for every $n \geq 2$. Show that $tr(A)$ belongs to F by following steps.

a Show that there is a polynomial $P(x) = \sum_i a_i x^i \in \bar{E}[x]$ with $a_0 = 1$ such that
$$\sum_i a_i tr(A^{i+k}) = 0, \qquad \forall k \geq 1.$$

b Show that we have a polynomial $Q = \sum_i b_i x^i \in F[x]$ with $b_0 = 1$ such that
$$\sum_i b_i tr(A^{i+k}) = 0, \qquad \forall k \geq 2.$$

c Let $t \in \bar{E}$ be an eigenvalue of A with multiplicity m invertible in F. Show that $Q(t) = 0$.

d Show that $tr(A)$ belongs to F.

Hint: Let $t_i \in \bar{E}$ be all distinct non-zero eigen values of A with multiplicity m_i invertible in F. Then
$$tr(A^n) = \sum_i m_i t_i^n.$$

2. a Prove that $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is not a UFD.

b Prove that $\mathbb{C}[x, y]/(x^2 + y^2 - 1)$ is a PID.